Introduction to computer-assisted proofs in nonlinear analysis

Séminaire doctorant du LAMFA

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Wednesday 5 February 2025

A first example

Let us compute $\sin(0)$ and $\sin(\pi)$ using Python.

Image from https://fr.wikipedia.org/wiki/Fichier:Python-logo-notext.svg

Floating-point numbers in a nutshell

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$\mathbb{F}$: set of finite 64 bit (double precision) floating-point numbers.

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- etc.

How not to launch a rocket

Figure: An Ariane 5 launcher (click for the video)

Image from https://commons.wikimedia.org/wiki/File:Ariane_5_with_James_Webb_Space_Telescope_Prelaunch_(51773093465).jpg,

video from https://www.youtube.com/watch?v=1qRUFg-Pte0

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To summarize:

A first (obvious) limitation of numerical computations

$\mathbb{F}$ is **finite**!

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There are thus several *rounding modes*, depending on whether the result is to be rounded up, down, towards zero, etc.

Accumulation of round-off errors

The Vancouver stock index

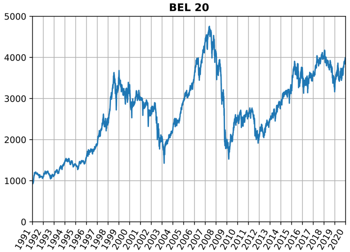

Figure: The BEL20 stock index

Image from https://commons.wikimedia.org/wiki/File:BEL_20.svg

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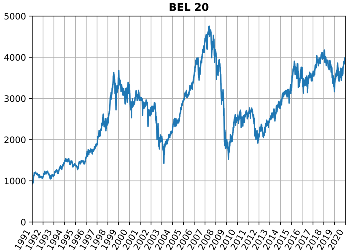
The Vancouver stock index

Between 1982 and 1983, the Vancouver stock index dropped anomalously due to the accumulation of small round-off errors, due to the fact that quantities were always rounded *down* after each computation.



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Patriot missiles

Figure: A Patriot missile launch

Image from

https://upload.wikimedia.org/wikipedia/commons/f/f8/Patriot_missile_launch_b.jpg

Accumulation of round-off errors
Patriot missiles

In 1991, American Patriot missiles failed to intercept an incoming Scud missile, killing 28 soldiers and injuring 100 other people, due to a bad computation of internal time due to an accumulation of round-off errors.

Figure: A Patriot missile launch

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Approximation errors are typically studied by numerical analysts: rigorous error bounds, convergence results, etc.

As for round-off errors, in "practical applications" it is important to **be aware** of them and to **keep them small by design**. This typically involves a suitable **stability analysis** of the numerical methods.

Where are we now?

For us, an important question remains.

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If only one could ignore round-off errors...

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- to physicists: physical measurements are performed up to a finite precision anyway.

Although this may seem a paradox, all exact science is dominated by the idea of approximation.

— Bertrand Russell, The Scientific Outlook

The class $\mathcal{I}_{\mathbb{R}}$ of intervals

The intervals we will consider are the topologically closed and connected subsets of $\mathbb{R}$ (as specified in the standard IEEE-1788 devoted to interval arithmetic[1]), i.e. they belong to the class $\mathcal{I}_{\mathbb{R}}$ of subsets of $\mathbb{R}$ defined by

$$\begin{aligned}\mathcal{I}_{\mathbb{R}} := \left\{\emptyset\right\} &\cup \left\{[a, b] \mid a, b \in \mathbb{R}, a \leq b\right\}\\ &\cup \left\{[a, +\infty[\mid a \in \mathbb{R}\right\}\\ &\cup \left\{]-\infty, b] \mid b \in \mathbb{R}\right\}\\ &\cup \left\{]-\infty, +\infty[:= \mathbb{R}\right\}.\end{aligned}$$

[1] See https://standards.ieee.org/ieee/1788/4431/.

Operations on intervals

Given two intervals $\mathbf{x}$ and $\mathbf{y}$, their *sum* is given by

$$\mathbf{x} + \mathbf{y} := \left\{ x + y \mid x \in \mathbf{x}, y \in \mathbf{y} \right\},$$

their *difference* by

$$\mathbf{x} - \mathbf{y} := \left\{ x - y \mid x \in \mathbf{x}, y \in \mathbf{y} \right\}$$

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Examples and surprises: on the blackboard!

In general: interval extensions

Definition

Let $D \subseteq \mathbb{R}$ be a set and let $F : D \to \mathbb{R}$ be a map.

An *interval extension* of F is an application $\mathbf{F} : \mathcal{I}_\mathbb{R} \to \mathcal{I}_\mathbb{R}$ which satisfies the *containment property*, namely so that for all $\mathbf{x} \in \mathcal{I}_\mathbb{R}$, the set

$$F(\mathbf{x}) := \left\{ F(x) \mid x \in \mathbf{x} \cap D \right\}$$

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Examples on the blackboard! *Compare extensions of $F : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ with the product operation.*

Fundamental theorem of interval arithmetic

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Allows to obtain interval extensions of complicated functions by composing interval extensions of its subparts.

In practice

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In practice, the implementation will use intervals from the set

$$\mathcal{I}_{\mathbb{F}} := \left\{ \mathbf{x} = [\underline{x}, \overline{x}] \mid \underline{x} \leq \overline{x} \text{ are two floating-point numbers} \right\} \cup \left\{ \emptyset \right\}.$$

Back to the computation of sin(π)

Let us use the "`mpmath`" library[2] in Python3 and ask the value of

`iv.pi`

then

`iv.sin(iv.pi)`.

[2] See in particular the module `iv`, devoted to interval arithmetic at https://www.mpmath.org/doc/1.0.0/contexts.html.

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  For instance, `iv.sin(x)` could return `[-1, 1]` regardless of the value of `x`, but this bound is useless.
- Nevertheless, it is in principle possible to show that given matrices are invertible, positive/negative definite... using interval arithmetic.

Locating roots of a function

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We may thus divide $[0,1]$ into many "small" intervals and discard all those for which we are sure that F has no roots, this being determined by evaluating the interval extension $\mathbf{F}$. We end up with (possibly many) small intervals such that all potential roots of F belong to one of those.

Application of interval arithmetic to nonlinear analysis

Existence of the Lorenz strange attractor

The system of ODEs

$$\begin{aligned} \partial_t x_1 &= -\sigma x_1 + \sigma x_2 \\ \partial_t x_2 &= \rho x_1 - x_2 - x_1 x_3, \\ \partial_t x_3 &= -\beta x_3 + x_1 x_2 \end{aligned}$$

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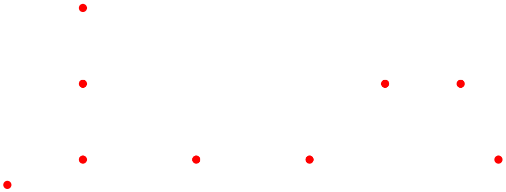
This fact, though conjectured since the 1960s, was only proved by Warwick Tucker in 1999, using a computer-assisted proof using interval arithmetic.

What is a compact metric graph?

A compact metric graph is made of a finite number of vertices



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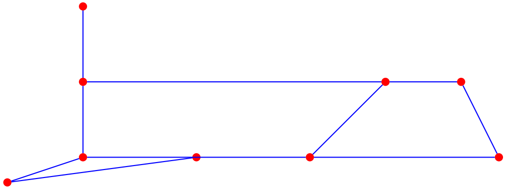

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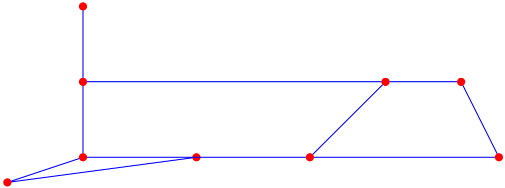
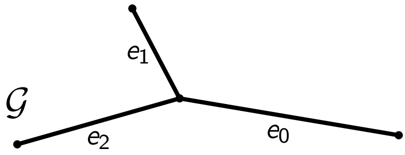


Metric graphs: the lengths of edges are important.

Functions defined on metric graphs



A compact metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3)

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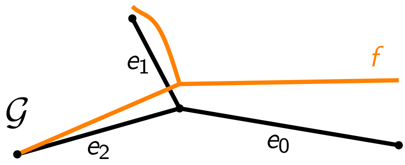

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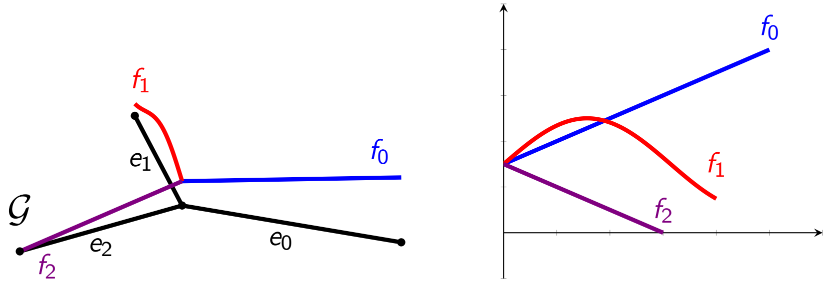
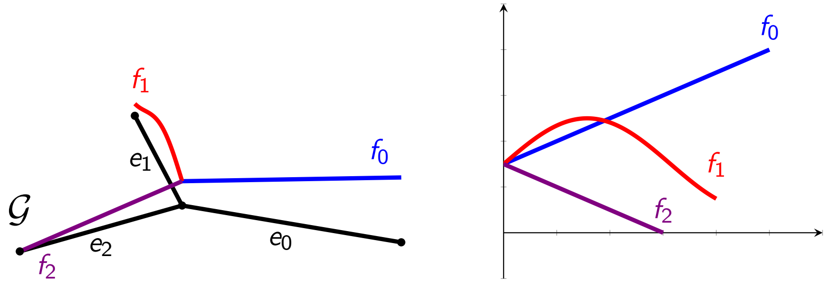


A compact metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3), a function $f : \mathcal{G} \to \mathbb{R}$, and the three associated real functions.

Functions defined on metric graphs



A compact metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3), a function $f : \mathcal{G} \to \mathbb{R}$, and the three associated real functions.

$$\int_{\mathcal{G}} f \,\mathrm{d}x := \int_0^5 f_0(x)\,\mathrm{d}x + \int_0^4 f_1(x)\,\mathrm{d}x + \int_0^3 f_2(x)\,\mathrm{d}x$$

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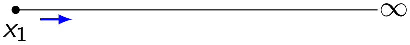
Remark: we always have $\dim E_1 = 1$ with $\gamma_1 = 0$, considering constant functions.

Kirchoff's condition: degree one nodes



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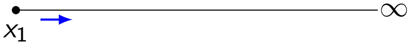
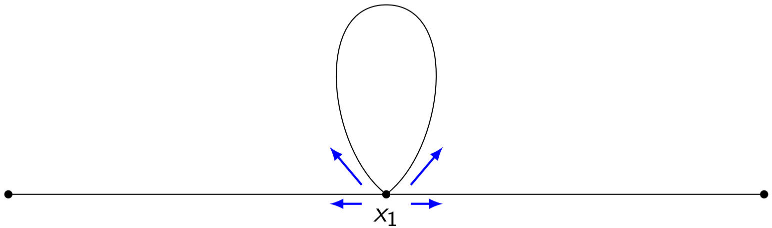


$$\lim_{\substack{t \to 0 \\ t>0}} \frac{u(x_1+t)-u(x_1)}{t} = 0$$

In other words, the derivative of u at x_1 vanishes: this is the usual Neumann condition.

Kirchoff's condition in general: outgoing derivatives



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Question

What about $p > 2$?

The quasilinear regime $p \approx 2$ ($p > 2$)

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$$\int_{\mathcal{G}} u_* \ln |u_*| \varphi \, \mathrm{d}x = 0 \qquad \forall \varphi \in E_2.$$

We say that $u_* \in E_2$ is a *solution of the reduced problem* if the above condition holds.

Variational formulation

The functional $\mathcal{J}_* : E_2 \to \mathbb{R}$

$$\mathcal{J}_*(\varphi) := \frac{1}{4} \int_{\mathcal{G}} \varphi^2(x) \big(1 - 2 \ln |\varphi(x)|\big) \, \mathrm{d}x$$

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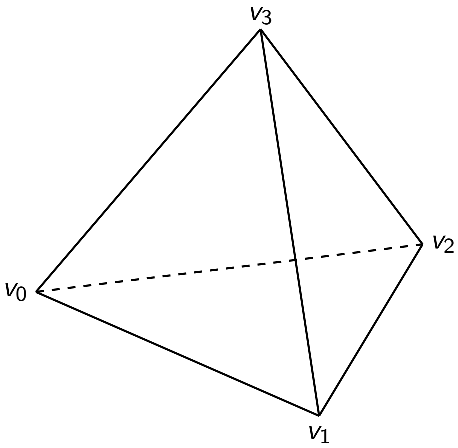
Using a “Lyapunov-Schmidt” argument, we can show **existence and uniqueness results around a nondegenerate critical point** for $(\mathcal{P}_p)$, when $p \approx 2$.

The tetrahedron

In the remainder of the talk, we will only consider the following graph $\mathcal{G}_{\mathrm{t}}$.



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In this way, we obtain an *isometric group action*

$$G_t \times E_2 \to E_2 : (g, \varphi) \mapsto g \cdot \varphi,$$

such that $J_*(g \cdot \varphi) = J_*(\varphi)$ for all $(g, \varphi) \in G_t \times E_2$.

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then u is a critical point of J.

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Question

Does $\mathcal{J}_*$ possess critical points other than the ones of the four aforementioned families?

A computer-assisted answer

Theorem (De Coster, G., Troestler (2024))

All critical points of $\mathcal{J}_* : E_2 \to \mathbb{R}$ (for the tetrahedron graph) belong to one of the four families obtained thanks to the symmetries.

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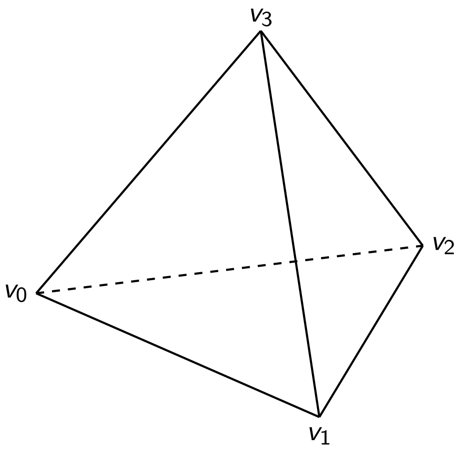
Things worked out!

Thanks for your attention!



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